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PRE-FRACTURE ZONE MODELING FOR AN INTERFACE CRACK IN AN ISOTROPIC BIMATERIAL

An interface crack in an infinite bimaterial space under remote combining loading is considered. The complex potentials approach is applied and an exact analytical solution of this problem is presented. To remove the oscillating singularities which occur in this solution a model based on the introduction of the shear stress pre-fracture zones at the crack tips is suggested. In the framework of this assumption the nonhomogeneous combined Dirichlet-Riemann boundary value problem with the conditions at infinity is formulated and its analytical solution is presented. The length of this zone is found from the condition of restriction of the shear stress at the end point of the zone. In this case the shear stress becomes finite in the right hand side of the pre-fracture zone while the normal stress has a square root singularity at the crack tip. The energy release rates (ERR) at the crack tip and also along the pre-fracture zone are found and their total values are compared the ERR of the classical model. For the case of a similar problem, but for finite sized body the finite element method is applied. The crack length is assumed to be much smaller then characteristic body size. The finite element net with two levels of concentration is constructed. The first level provides the uniform concentration from the boundaries of the body to the crack and the second level assumes the similar concentration at the singular points of the pre-fracture zone. The different values of the shear stress in the pre-fracture zone are considered and the local ERR at the singular points as well as the global energy release rate is found. It is shown that for different values of the mentioned shear stress the global energy release rate remains almost invariable and is in a a good agreement with the analytical solution.

Keywords: Analytical solution, interface crack, pre-fracture zone, finite element method.

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ЗОНИ ПЕРЕДРУЙНУВАННЯ ДЛЯ МІЖФАЗНОЇ ТРІЩИНИ В ІЗОТРОПНОМУ БІМАТЕРІАЛІ

Розглянута тріщина між двома ізотропними матеріалами під дією віддаленого комбінованого навантаженням. Для усунення осцилюючої особливості, що виникає при використані класичної моделі тріщини пропонується підхід, оснований на введенні зсувних зон передруйнування на її продовженнях. Проведено аналітичне дослідження цієї моделі шляхом зведення поставленої проблеми до комбінованої задачі лінійного спряження Діріхле-Рімана. Завдяки точному розв’язку цієї задачі знайдені досить прості вирази для напружень, їх коефіцієнтів інтенсивності та швидкості звільнення енергії. Розгляну також аналогічна модель для тіла скінчених розмірів при умові, що розмір тріщини значно менший характерного розміру області. Розглянуті різні величини зсувного напруження у зонах передруйнування. У цьому випадку розв’язок побудовано методом скінчених елементів. Знайдено локальні швидкості звільнення енергії біля сингулярних точок, а також її глобальні значення. Виявлено хорошу узгодженість аналітичного та чисельного розв’язків.

Ключові слова: Аналітичний розв’язок, метод скінчених елементів, тріщина між двома матеріалами, зона передруйнування.

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ЗОНЫ ПРЕДРАЗРУШЕНИЯ ДЛЯ МЕЖФАЗНОЙ ТРЕЩИНЫ В ИЗОТРОПНОМ БИМАТЕРИАЛЕ

Рассмотрена трещина между двумя изотропными материалами под действием удаленной комбинированной нагрузки. Для устранения осциллирующей особенности, которая возникает при использовании классической модели трещины предлагается подход, основанный на введении сдвиговых зон предразрушения на ее продолжениях. Проведено аналитическое исследование этой модели путем сведения поставленной проблемы в комбинированной задачи линейного сопряжения Дирихле-Римана. Благодаря точному решению этой задачи найдены достаточно простые выражения для напряжений, их коэффициентов интенсивности и скорости освобождения энергии. Рассмотрю также аналогичная модель для тела конечных размеров при условии, что размер трещины значительно меньше характерного размера области. Рассмотрены различные величины касательногонапряжения в зонах предразрушения. В этом случае решение построено методом конечных элементов. Найдено локальные скорости освобождения энергии возле сингулярных точек, а также ее глобальные значения. Выявлено хорошую согласованность аналитического и численного решений.

Ключевые слова: Аналитическое решение, метод конечных элементов, трещина между двумя материалами, зона предразрушения.

Introduction. An interface crack investigation is of paramount importance for various devices of piezoelectric composite materials manufacture and service because such cracks are very often the main reason of the constructions failure. Interface cracks between isotropic materials has been actively studying for the last years. Particularly oscillating interface crack model has been developed by Williams [1] and reexamined by Rice [2]. To eliminate physically unreal oscillating singularity at the crack tips the contact interface crack model has been suggested by Comninou [3] and developed later in numerical and analytical manner in the papers [4]-[6].

Usually the interface between different materials is softer than the adhered components, therefore, thin pre-fracture zones develop at the crack tips along the interface. Such zones were taken into account in paper [7] with use of dry friction law and in [8], [9] by means of applyingLeonov-Panasyuk model [10] for the simulation of the mechanical fields in these zones. However it is known [4, 6] that shear stress along the interface is the most dangerous at the tip of an interface crack considered in the framework of the contact zone model. Therefore, the consideration of pre-fracture zones based upon ultimate shear stress is rather pertinently. Such approach concerning a crack between two anisotropic materials was developed in the paper [11], however the energetic consideration has not got due attention in this paper.

In the present paper an interface crack between two isotropic materials with ultimate shear stress pre-fracture zones is studied. An analytical method is applied and the solution free from an oscillation is obtained. By means of deformation energy consideration the comparison of this solution with the associated results obtained by finite element method for a finite sized composite with an interface crack is carried out and good agreement is revealed.

Formulation of the basic relations. A bimaterial composed of two different isotropic semi-infinite spaces $y > 0$ and $y < 0$ with different mechanical characteristics is considered .The stresses (σ_{ij}) and the displacements (u_i) for a two-dimensional problem of the elasticity can be presented in the form [12]

$$\sigma_{22}^{(k)} - i\sigma_{12}^{(k)} = \Phi_k(z) + \overline{\Phi_k(z)} + z\overline{\Phi_k'(z)} + \overline{\Psi_k(z)}\,; \quad \text{(1a)}$$

$$2\mu_k(u_1^{(k)} + iu_2^{(k)}) = \kappa_k\phi_k(z) - z\overline{\Phi_k(z)} - \overline{\psi_k(z)}\,, \quad \text{(1b)}$$

where $\Phi_k(z) = \phi_k'(z)$ and $\Psi_k(z) = \psi_k'(z)$ are analytical functions of the complex variable $z = x + iy$ in the respective half-planes. Besides,

$$\kappa_k = \begin{cases} 3-4\nu_k & \text{(for plane strain);} \\ \dfrac{3-\nu_k}{1+\nu_k} & \text{(for plane stress),} \end{cases} \quad (2)$$

μ_k and ν_k are the shear moduli and Poisson's ratios, respectively. A bar denotes the conjugation operation, and a prime denotes the differentiation with respect to the argument, $k = 1{,}2$ are related to the upper and lower half-planes, respectively.

By introducing the following analytical functions in the respective half-planes according to

$$\omega_k(z) = z\overline{\phi}_k'(z) + \overline{\psi}_k(z)\,,$$

and replacing z by $\overline{z}$ gives

$$\overline{\psi}_k(\overline{z}) = \omega_k(\overline{z}) - \overline{z\phi_k'(z)}; \quad \text{(3a)}$$

$$\overline{\psi_k'(z)} = \omega_k'(\overline{z}) - \overline{\phi_k'(z)} - \overline{z\phi_k''(z)}. \quad \text{(3b)}$$

Substituting (3) into (1) and denoting $\omega_k'(z) = \Omega_k(z)$ leads to the following expressions

$$\sigma_{22}^{(k)} - i\sigma_{12}^{(k)} = \Phi_k(z) + (z-\overline{z})\overline{\Phi_k'(z)} + \Omega_k(\overline{z})\,; \quad \text{(4a)}$$

$$2\mu_k(u_1^{(k)} + iu_2^{(k)}) = \kappa_k\phi_k(z) + (\overline{z}-z)\overline{\Phi_k(z)} - \omega_k(\overline{z})\,. \quad \text{(4b)}$$

Satisfying the continuity condition

$$\sigma_{22}^{(1)} - i\sigma_{12}^{(1)} = \sigma_{22}^{(2)} - i\sigma_{12}^{(2)} \quad \text{for } x \in (-\infty,\infty)\,,$$

by using the expression (4a), gives

$$\Phi_1^+(x) - \Omega_2^+(x) = \Phi_2^-(x) - \Omega_1^-(x) \quad \text{for } x \in (-\infty,\infty)\,. \quad (5)$$

The superscripts «+» and «-» denote here the limit values of the analytical functions for $y \to +0$ and $y \to -0$, respectively.

Since the two sides of (5) represent the limit values of two analytical functions in the respective half-planes, therefore, both functions can be analytically extended into the entire plane. Using the condition that the functions are bounded at infinity the equation (5) yields

$$\begin{cases} \Phi_1(z) = \Omega_2(z) & \text{for } y > 0; \\ \Phi_2(z) = \Omega_1(z) & \text{for } y < 0. \end{cases} \quad (6)$$

Introducing a new function

$$F(z) = \begin{cases} \dfrac{\kappa_1}{2\mu_1}\Phi_1(z) + \dfrac{1}{2\mu_2}\Omega_2(z) & \text{for } y > 0; \\ \dfrac{\kappa_2}{2\mu_2}\Phi_2(z) + \dfrac{1}{2\mu_1}\Omega_1(z) & \text{for } y < 0 \end{cases} \quad (7)$$

and by using the relations (6) and (7) one obtains the following equations

$$\sigma_{22}^{(1)}(x) - i\sigma_{12}^{(1)}(x) = g\left\{F^+(x) + \gamma F^-(x)\right\}; \quad (8)$$

$$[u_1'(x)] + i[u_2'(x)] = F^+(x) - F^-(x)\,, \quad (9)$$

where

$$g = \frac{2\mu_1\mu_2}{\kappa_1\mu_2 + \mu_1}; \quad \gamma = \frac{\kappa_1\mu_2 + \mu_1}{\kappa_2\mu_1 + \mu_2}\,. \quad (10)$$

It is clear from the equation (9) that the function $F(z)$ is analytic in the whole plane with a cut along $(-\infty,\infty)\backslash L$. The expressions (6) and (14), (15) play an important role in the following analysis, and moreover, they can by used for the solution of variousmechanical problems for a bimaterial plane with cuts along the interface.

A crack at the interface of two materials. We assume that the crack is situated at the section $[c, a]$ of the material interface between two semi-infinite spaces. The half-spaces are loadedby uniformly distributed normal stress $\sigma^{\infty}{}_{22}$ and the shear stress $\sigma^{\infty}{}_{12}$ at infinity, which do not depend on coordinate x_3. The crack faces are free of loading. Such external stresses call plane deformation state, therefore, only the cross-section orthogonal to x_3 (Fig. 1) can be considered (the zone $[a, b]$ is not need for a while and will be used later).

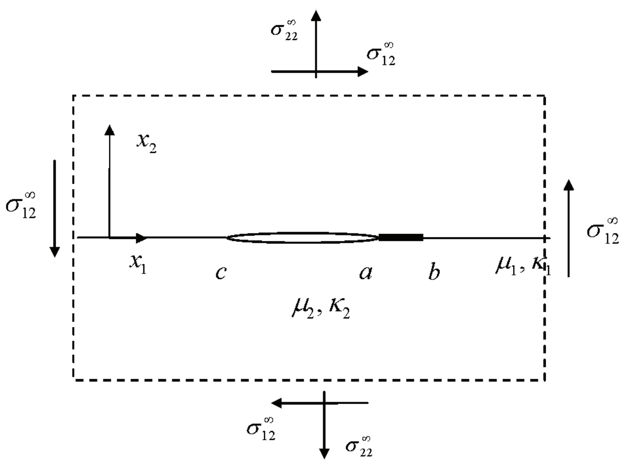

Figure 1 – A crack between two isotropic materials

The boundary conditions for the formulated problem are the following

$$\sigma_{21}^{(1)}=\sigma_{21}^{(2)}=0\,;\quad \sigma_{22}^{(1)}=\sigma_{22}^{(2)}=0 \text{ for } c<x_1<a;$$
$$\langle\sigma_{21}\rangle=0\,;\ \langle\sigma_{22}\rangle=0\,;\ \langle u_1'\rangle=0\,;$$
$$\langle u_2'\rangle=0 \text{ for } x_1\notin(c,a)\,. \tag{11}$$

The solution of this problem under the condition at infinity

$$F_1(z)\big|_{z\to\infty}=\tilde\sigma_{22}-i\tilde\sigma_{21}\,, \tag{12}$$

where $\tilde\sigma_{22}=\dfrac{\sigma_{22}^{\infty}}{r_1}$; $\tilde\sigma_{21}=\dfrac{\sigma_{21}^{\infty}}{r_1}$; $r_1=(1+\gamma_1)g$

was obtained with use of [12] in the form

$$F(z)=(\tilde\sigma_{22}-i\tilde\sigma_{21})\frac{z-(a+c)/2-i\varepsilon l}{\sqrt{(z-c)(z-a)}}\left(\frac{z-c}{z-a}\right)^{i\varepsilon}, \tag{13}$$

where $\varepsilon=\dfrac{1}{2\pi}\ln\gamma_1$; $l=b-c$.

The stresses at the interface are obtained from (8), (13) as follows

$$\sigma_{22}^{(1)}(x_1,0)-i\sigma_{21}^{(1)}(x_1,0)=\left(\sigma_{22}^{\infty}-i\sigma_{21}^{\infty}\right)\times$$
$$\frac{x_1-(a+c)/2-i\varepsilon l}{\sqrt{(x_1-c)(x_1-a)}}\left(\frac{x_1-c}{x_1-a}\right)^{i\varepsilon} \text{ for } x_1>a \tag{14}$$

and the derivative of the displacement jumps are found from the formula (9) in the form

$$\langle u_1'(x_1,0)\rangle+i\langle u_2'(x_1,0)\rangle=$$
$$=-\frac{\left(\sigma_{22}^{\infty}i+\sigma_{21}^{\infty}\right)}{g\sqrt{\gamma_1}}\frac{\left(x_1-(a+c)/2-i\varepsilon l\right)}{\sqrt{(x_1-c)(a-x_1)}}\left(\frac{x_1-c}{a-x_1}\right)^{i\varepsilon}$$
$$\text{for } c<x_1<a. \tag{15}$$

Integrating the last relation, we obtain

$$\langle u_1(x_1,0)\rangle+i\langle u_2(x_1,0)\rangle=$$
$$=\sqrt{(x_1-c)(a-x_1)}\left\{\frac{\left(\sigma_{22}^{\infty}i+\sigma_{21}^{\infty}\right)}{g\sqrt{\gamma_1}}\left(\frac{x_1-c}{a-x_1}\right)^{i\varepsilon}\right\}$$
$$\text{for } c<x_1<a. \tag{16}$$

The energy release rate (ERR) can be found by use of the formulas (14), (16) and can be presented in the form [13]

$$G_{osc}=\frac{q}{4}a\pi(1+4\varepsilon^2)(\sigma^2+\tau^2)\,, \tag{17}$$

where $q=\dfrac{(\mu_1+\mu_2\kappa_1)(\mu_2+\mu_1\kappa_2)}{\mu_1\mu_2(\mu_1+\mu_2+\mu_2\kappa_1+\mu_1\kappa_2)}$.

Formulation of interface crack model free from oscillation. It follows from equations (14), (15) that the normal and shear stresses and both displacement jumps have an oscillating singularity at the crack tips. To transform this singularity to the non-oscillating onewe introduce the pre-fracture zones with constant shear stresses on the crack continuations. Because these zones are either both very short or one zone is substantially shorter than another one, we'll consider for simplicity only longer zone, assuming that it occurs at the right crack tip (Fig. 2). Simple transformation of half-spaces should be done if the longer zone occurs at another crack tip.

Thus the boundary conditions for the considered model are the following

$$\sigma_{21}^{(1)}=\sigma_{21}^{(2)}=0\,;\ \sigma_{22}^{(1)}=\sigma_{22}^{(2)}=0 \text{ for } c<x_1<a, \tag{18}$$
$$\sigma_{21}^{(1)}=\sigma_{21}^{(2)}=\tau_0\,;\ \langle\sigma_{22}\rangle=0\,;\ \langle u_2'\rangle=0 \text{ for } a<x_1<b, \tag{19}$$
$$\langle\sigma_{21}\rangle=0\,;\ \langle\sigma_{22}\rangle=0\,;\ \langle u_1'\rangle=0\,;\ \langle u_2'\rangle=0$$
$$\text{for } x_1\notin(c,a)\,. \tag{20}$$

where τ_0 is the ultimate shear stress occurs at the crack continuation in the pre-fracture zone [a, b]. It can be for example the shear yield stress for the softer matrix or for the adhesive.

Satisfying the interface conditions (18) with use of (7) we get the equation.

$$F_1^{+}(x_1)+\gamma_1F_1^{-}(x_1)=0 \quad \text{for } c<x_1<a. \tag{21}$$

Besides the conditions (19 and (20) lead to the equations

$$\mathrm{Im}\left[F_1^{+}(x_1)+\gamma_1F_1^{-}(x_1)\right]=-\tau_s/g\,;$$
$$\mathrm{Im}\left[F_1^{+}(x_1)-F_1^{-}(x_1)\right]=0 \quad \text{for } a<x_1<b. \tag{22}$$

Satisfaction of the boundary conditions (20) provides the analyticity of the function $F(z)$ outsides of the interval (c, b). The relations (22) lead to the equation

$$\mathrm{Im}\,F_1^{\pm}(x_1)=\tau \quad \text{for } a<x_1<b. \tag{23}$$

where $\tau=-\tau_0/r_1$; $r_1=(1+\gamma_1)g$.

The Eqs. (21) and (23) present the nonhomogeneous combined Dirichlet-Riemann boundary value problem. The conditions at infinity (12) are valid for this problem as well.

With use of Nahnein and Nuller [14] the general solution of this problem was presented in [11] in the form

$$F_1(z)=\left(P(z)+\Phi(z)\right)X_1(z)+Q(z)X_2(z)\,, \tag{24}$$

where

$$P(z)=C_1z+C_2\,;\qquad Q(z)=D_1z+D_2\,;$$
$$X_1(z)=ie^{i\chi(z)}/\sqrt{(z-c)(z-b)}\,;$$
$$X_2(z)=e^{i\chi(z)}/\sqrt{(z-c)(z-a)}\,;$$
$$\chi(z)=2\varepsilon\ln\frac{\sqrt{(b-a)(z-c)}}{\sqrt{l(z-a)}+\sqrt{(a-c)(z-b)}}\,;\qquad z=x_1+ix_2\,;$$
$$\Phi(z)=\frac{\tau}{\pi}\left[-iY(z)L_1(z)+L_2(z)\right];$$
$$L_1(z)=\int_a^b\sqrt{\frac{t-c}{t-a}}\frac{\sinh\chi_0(t)}{t-z}dt\,;$$
$$L_2(z)=\int_a^b(t-c)(b-t)\frac{\cosh\chi_0(t)}{t-z}dt\,.$$
$$C_1=-\tilde\sigma_{23}\sin\beta-\tilde E_1\cos\beta\,;\quad D_1=\tilde\sigma_{23}\cos\beta-\tilde E_1\sin\beta\,,$$
$$C_2=-\frac{c+b}{2}C_1-\beta_1D_1\,;\ D_2=\beta_1C_1-\frac{c+a}{2}D_1+R\,,$$
$$R=\frac{\tau}{\pi}\int_a^b\sqrt{\frac{t-c}{t-a}}\sinh\varphi_0(t)dt\,;\quad \beta=\varepsilon\ln\frac{1-\sqrt{1-\lambda}}{1+\sqrt{1-\lambda}}\,;$$
$$\beta_1=\varepsilon\sqrt{(a-c)(b-c)}\,;\qquad \lambda=\frac{b-a}{b-c}. \tag{25}$$

Pre-fracture zone length and the energy release rate (ERR). According to Eqs. (8), (24) the stress field on the right from pre-fracture zone can be presented in the form:

$$\sigma_{22}^{(1)}(x_1,0)-i\sigma_{21}^{(1)}(x_1,0)=r_1F_1\left(x_1\right)=$$
$$=r_1\left\{\left(P\left(x_1\right)+\Phi(x_1)\right)X_1\left(x_1\right)+Q\left(x_1\right)X_2\left(x_1\right)\right\}. \quad (26)$$

For an arbitrary position of the point b right hand side of (25) is singular for $x_1\to b+0$. To remove this singularity the equation

$$P(b)+\Phi(b)=0.$$

should be satisfied. After some transformation this equation can be written in the form

$$-\delta\cos\beta-\sin\beta-2\varepsilon\sqrt{1-\lambda}(\cos\beta-\delta\sin\beta)+$$
$$+\frac{2}{\pi(b-c)}\frac{\tau_0}{\sigma_{22}^{\infty}}\int_a^b\sqrt{\frac{t-c}{b-t}}\cosh\varphi_0(t)dt=0, \quad (27)$$

where $\delta=\sigma_{21}^{\infty}/\sigma_{22}^{\infty}$ and $l=b-c$.

The obtained equation should be solved with respect to λ and then point b can be found from Eq. (25). As a rule the Eq. (27) can be solved numerically that permits to find the root of this equation which we denote λ_0.

The normal stress at (a, b) according to (8), (24) can be found in the form

$$\sigma_{22}^{(1)}(x_1,0)=t_1\left[F_1^+\left(x_1\right)+\gamma_1F_1^-\left(x_1\right)\right]-iE_b.$$

Substituting the formula (24), taking into account that [15]

$$X_1^{\pm}(x_1)=\frac{\pm e^{\pm\phi_0(x_1)}}{\sqrt{(x_1-c)(b-x_1)}};$$

$$X_2^{\pm}(x_1)=\frac{e^{\pm\phi_0(x_1)}}{\sqrt{(x_1-c)(x_1-a)}} \text{ for } x_1\in(a,b)$$

and applying Plemeli formulas [12] we arrive to the following expression

$$g^{-1}\sigma_{22}^{(1)}(x_1,0)=P(x_1)\frac{e^{\chi_0(x_1)}-\gamma_1e^{-\chi_0(x_1)}}{\sqrt{(x_1-c)(b-x_1)}}+$$
$$+Q(x_1)\frac{e^{\chi_0(x_1)}+\gamma_1e^{-\chi_0(x_1)}}{\sqrt{(x_1-c)(x_1-\mathrm{a})}}+$$
$$+\frac{\tau}{\pi}\left\{-\sqrt{\frac{x_1-a}{x_1-c}}\left[e^{\chi_0(x_1)}+\gamma_1e^{-\chi_0(x_1)}\right]L_1(x_1)+\right.$$
$$\left.+\frac{e^{\chi_0(x_1)}-\gamma_1e^{-\chi_0(x_1)}}{\sqrt{(x_1-c)(b-x_1)}}L_2(x_1)\right\}, \quad (28)$$

where the integrals $L_1(x_1)$ and $L_2(x_1)$ should be considered in sense of principal value on Cauchy [12].

The derivatives of the crack faces displacement jump at the interval (c, a) can be foundon the base of (24) with use of (9) in the form

$$\left\langle u_2'(x_1,0)\right\rangle=\mathrm{Im}\left[F_1^+\left(x_1\right)-F_1^-\left(x_1\right)\right].$$

Substituting the expression (24) one gets

$$\left\langle u_1'(x_1,0)\right\rangle+i\left\langle u_2'(x_1,0)\right\rangle=\frac{\gamma_1+1}{\sqrt{\gamma_1}}\left[\frac{P\left(x_1\right)+\Phi\left(x_1\right)}{\sqrt{b-x_1}}-\right.$$
$$\left.i\frac{Q\left(x_1\right)}{\sqrt{a-x_1}}\right]\frac{\exp\left[i\varphi^*\left(x_1\right)\right]}{\sqrt{x_1-c}} \text{ for } c<x_1<a, \quad (29)$$

where $\varphi^*\left(x_1\right)=2\varepsilon\ln\dfrac{\sqrt{(b-a)(x_1-c)}}{\sqrt{l\left(a-x_1\right)}+\sqrt{(a-c)(b-x_1)}}$

and the displacement jump can be found as

$$\left\langle u_1\left(x_1,0\right)\right\rangle+i\left\langle u_2\left(x_1,0\right)\right\rangle=$$
$$\int_c^x\left(\left\langle u_1'(t,0)\right\rangle+i\left\langle u_2'(t,0)\right\rangle\right)dt. \quad (30)$$

According to [15] the local energy release rate (ERR) at the point a can be presented in the form

$$G_a=\lim_{\Delta l\to0}\frac{1}{2\Delta l}\left\{\int_a^{a+\Delta l}\sigma_{22}^{(1)}\left(x_1,0\right)\left\langle u_2\left(x_1-\Delta l,0\right)\right\rangle dx_1\right.. \quad (31)$$

On the base of the formulas (28), (29) the following asymptotic forms are valid in the vicinity of the point a:

$$g^{-1}\sigma_{22}^{(1)}(x_1,0)\Big|_{x_1\to a+0}=Q(a)\frac{e^{\chi_0(a)}+\gamma_1e^{-\chi_0(a)}}{\sqrt{(a-c)(x_1-\mathrm{a})}};$$

$$\left\langle u_2'\left(x_1,0\right)\right\rangle\Big|_{x_1\to a-0}=-\frac{2\sqrt{\alpha}Q(a)}{\sqrt{(a-c)(\mathrm{a}-x_1)}}. \quad (32)$$

Performing further the integration in (31) with use of the value of integral [16]

$$\int_0^1\left(\frac{1-\tau}{\tau}\right)^{0.5-i\varepsilon}d\tau=\frac{\pi}{2}\frac{1-2i\varepsilon}{\sqrt{\alpha}},$$

and taking into account $e^{\chi_0(a)}+\gamma_1e^{-\chi_0(a)}=2\sqrt{\gamma_1}$, leads to

$$G_a=\pi r_1Q^2(a)/(a-c). \quad (33)$$

Following the approach of Gao *et al.* [17], we use a contour Γ closing the section (a, b). In this case

$$J_Y=\oint_\Gamma\left\{Wn_1-\sigma_{ij}n_iu_{j,i}\right\}ds,$$

where W is the specific energy of deformation. Using the path-independent property of J_Y, we tend the contour Γ to the section (a, b). Taking into account that the thickness of this zone tends to zero we arrive at the formula

$$J_Y=\tau_0\left\langle u_1\left(a,0\right)\right\rangle, \quad (34)$$

where $\left\langle u_1\left(a,0\right)\right\rangle$ is the tangential crack faces jump at the point a.

Numerical analysis and discussion. The numerical analysis was performed for the interface crack with $c=-10$ mm; $a=9.8$ mm; $b=10$ mm and $E_1=3\times10^4$ N/mm^2; $\nu_1=0.29$. For convenience without the loss of generality as the lower material was chosen the artificial one with the characteristics $E_2=3\times10^8$ N/mm^2; $\nu_2=0.345$. Different values of τ_0 were chosen and different values of the external mechanical loadings were considered. The results of the calculations of the right point of the yield zone b, $\left\langle u_1\left(a,0\right)\right\rangle$, the values of the ERR for the open crack model G_{osc}, the values of G_a, and the total ERR

$$G=G_a+J_y \quad (35)$$

are presented in Table 1 for $\sigma_{22}^{\infty}=10$ N/mm^2 and different values of σ^{∞}_{12}.

It is clearly seen from the two last columns of the Table 1 that the values of G, obtained for the developed model and the ERR of the classical (oscillating) model are in good agreement and the differences do not exceed 2 %. It is interesting to note also that the input of J_y in G is rather sensitive for $\tau_0 \neq 0$.

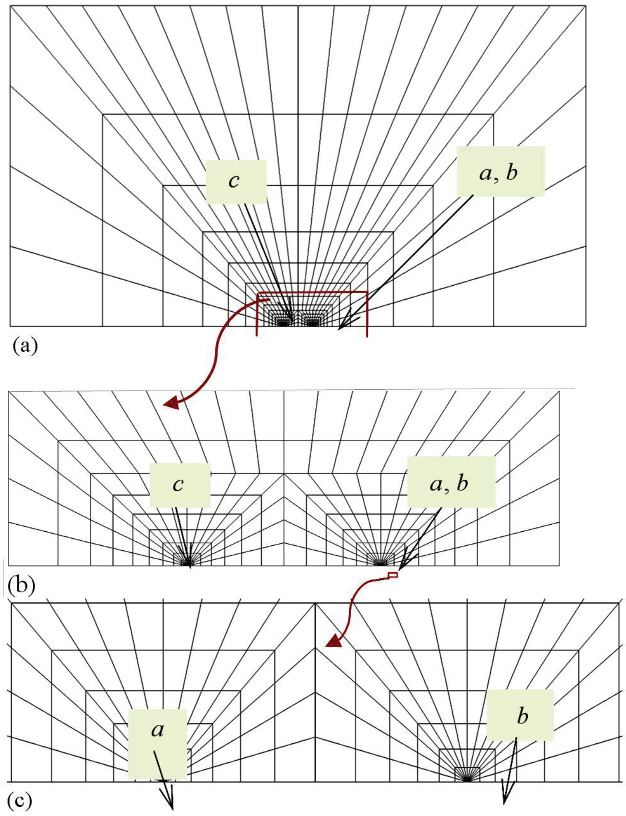

Figure 2 – Global mesh at the whole region(a) and local meshes in the vicinity of the crack region (b) and at the right crack tip (c)

For the conformation of the validity of the obtained analytical formulas and results the finite element simulation with use of the student version of ABAQUS were carried out. In this case the lower material was assumed to be absolutely rigid and the upper is the same as earlier. Moreover, the finite region with the dimension much larger than the crack length was considered. Therefore, the interface crack with the above mentioned characteristics along the fixed end of the rectangle $-200\text{ mm} \le x_1 \le 200\text{ mm}$; $0\text{ mm} \le x_2 \le 200\text{ mm}$ was studied.

The finite elements net with the local zones of the net refinements are shown in Fig. 2.

The results of the FEM calculations for $\sigma_{22}^{\infty} = 10\,\text{N/mm}^2$, different values of τ_0 as well as $\sigma^{\infty}_{12} = 0$ and $\sigma_{12}^{\infty} = -4\ \text{N/mm}^2$ are given in Tables 2 and 3, respectively.

In these Tables the local ERRs G_a and G_b at the points a and b, respectively, are presented. Besides, the tangential jump of displacement $\langle u_1(a,0)\rangle$, which permits to calculate the ERR J_Y on the formula (34), is also given. Finally the value of the ERR

$$G = G_a + G_b + J_y \tag{36}$$

is also presented. It should be mentioned that for $\lambda = \lambda_0$, i.e. for real pre-fracture zone length, the formula (36) transforms into the formula (35).

Table 1 – The values of the geometrical parameters and the energy release rates for $\sigma_{22}^{\infty} = 10\,\text{N/mm}^2$ and different values of σ^{∞}_{12} and τ_0. In this and the following Tables dimension of stresses is N/mm^2, lengths and displacements – mm, ERR – N/mm

σ_{12}^{∞}	τ_0	$\langle u_1(a,0)\rangle \times 10^4$	b	$G_a \times 10^2$	$G \times 10^2$	$G_{osc} \times 10^2$
0	-30.1	4.19	9.9997	3.26	4.52	4.54
-4	-59.1	5.99	10.0011	1.78	5.32	5.22
-8	-88.5	7.75	10.0001	0.738	7.59	7.45

Table 2 – The values of the fracture mechanical parameter for $\sigma^{\infty}_{12} = 0$ and different valuesof the shear stress in the pre-fracture zone

τ_0 [N/mm^2]	$\langle u_1(a,0)\rangle \times 10^4$	$G_a \times 10^2$	$G_b \times 10^2$	$G \times 10^2$
0	6.46	3.92	0.675	4.59
20	4.91	3.52	0.081	4.58
25	4.46	3.42	0.0226	4.59
27	4.45	3.38	0.0094	4.60
29	4.30	3.35	0.00089	4.59
30.1	4.19	3.33	≈ 0	4.60

Table 3 – The values of the fracture mechanical parameter for $\sigma_{12}^{\infty} = -4\ \text{N/mm}^2$ and different values of the shear stress in the pre-fracture zone

τ_0 [N/mm^2]	$\langle u_1(a,0)\rangle \times 10^4$	$G_a \times 10^2$	$G_b \times 10^2$	$G \times 10^2$
0	10.31	2.77	2.57	5.34
30	8.08	2.27	0.638	5.33
35	7.70	2.20	0.442	5.34
45	6.96	2.04	0.157	5.33
52	6.44	1.94	0.0437	5.33
59.1	5.91	1.84	≈ 0	5.33

It seen from the Tables 2 and 3, that the value of τ_0 essentially influences the magnitudes of G_a, G_b and $\langle u_1(a,0)\rangle$ i.e. J_Y. However the total ERR G remains almost invariant with respect to the value of τ_0, which acts in the pre-fracture zone and plays a role of an ultimate stress. Also the results in the last lines of Tables 2, 3 are in good agreement with the associated analytical results presented in Table 1. This fact confirms the correctness of the obtained solutions.

Conclusion. A crack at the interface of two dissimilar semi-infinite spaces under inplane normal-shear field of stresses is studied. The presentations of the stresses and the derivatives of the displacement jumps via sectional-analytic function are used. On the basis of these presentations an exact analytical solution of the formulated problem possessing an oscillating singularity is given. To eliminate this singularity a model based on the introduction of the shear stress pre-fracture zones at the crack tips is suggested. The mathematical simulation of this model leads to the nonhomogeneous combined Dirichlet-Riemann boundary value problem (21), (23), which was solved analytically. The length of pre-fracture zone is defined from the condition of restriction of the shear stress at its boundary point that leads to the simple transcendental equation (27).The local energy release rates (ERR) at the crack tip and also the ERR along the pre-fracture zone are found. Their summary value which is called global or total ERR is compared with the ERR of the classical model and their good agreement is found out. A similar problem for an interface crack in a finite sized body was considered by the finite element method with use of the student version of ABAQUS. It was assumed that size of the body is essentially larger than the crack length. In this case the results of the finite element method calculations demonstrate a good agreement with the analytical solution that confirmed the correctness of the developed approach and the obtained solution.

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