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I. MORACHKOVSKA, G. TIMCHENKO, E. LYUBITSKAYA

RESEARCH OF ELASTO-PLASTIC BENDING OF THIN SHELLS AND PLATES BY THE R-FUNCTIONS METHOD

Розглядається ефективний метод розв'язання нелінійних крайових задач пружно-пластичного згину тонких пологих оболонок який базується на теорії R-функцій. Задача зводиться до знаходження точок стаціонарності запропонованих змішаних варіаційних функціоналів, лінеаризованих за схемою методу послідовних навантажень і Ньютона-Канторовича спільно з методом змінних параметрів пружності. Чисельні дослідження виконані з використанням програмуючої системи «ПОЛЕ». Встановлено нові закони нелінійного деформування пологих оболонок і пластин складної форми в плані.

Ключові слова: тонкі пологі оболонки, пружно-пластичні деформації, теорія R-функцій.

Рассматривается эффективный метод решения нелинейных краевых задач упруго-пластического изгиба тонких пологих оболочек базирующийся на теории R-функций. Задача сводится к нахождению точек стационарности, предложенных смешанных вариационных функционалов, линеаризованных по схеме метода последовательных нагружений и Ньютона-Канторовича совместно с методом переменных параметров упругости. Численные исследования выполнены с использованием программирующей системы «ПОЛЕ». Установлены новые законы нелинейного деформирования пологих оболочек и пластин сложной формы в плане.

Ключевые слова: тонкие пологие оболочки, упруго-пластические деформации, теория R-функций.

The effective method basing on theory of R-functions and variational structural method is developed for solving non-linear boundary problems. Elastic-plastic bending of thin shallow shells is considered. The problems are reduced to finding stationary points of suggested mixed variational functionals according to the initial linearization due to usage of subsequent loading and Newton-Kantorovich jointly with method of varying elastic parameters. The method is used for automatic calculations in «POLE» programming system for investigations of shell structural elements. The numerical justification of the method is given. New laws of non-linear deformation of shallow shells and plates with complex shapes in plane are established.

Keywords: shallow shells, elasto-plastic deformations, R-functions theory.

Introduction

Many of the technology problems associated with the deformation of a thin shell, and this explains the development of a geometrically and physically nonlinear theory of shells, with the development of methods of research of stress strain state of shell structures, operating beyond the limits of elasticity [1-3].

Mathematical problems of elasto-plastic deformation of flexible membranes are formulated for non-linear differential equations under certain boundary and initial conditions. Mathematical methods, allowing to explore and find solutions of nonlinear differential equations, quite complicated. This paper proposes algorithms and some results of the solution of such problems on the basis of the known variational-structural method and the theory of R-functions [4-7]. This method became widespread in the international scientific literature under the name abbreviaturny RFM. For the first time the RFM applied to the solution of geometrically nonlinear problems of theory of plates and shells in [2] and its further development are presented in the review [6,7].

1 Mathematical Problem Formulation

Consider a thin shallow shell of constant thickness h under a transverse load. The mathematical formulation of the problem is performed via theory of small elasto-plastic deformations in the form of the method of variable elasticity parameter. Then the equations of state for the problems of bending of thin shallow shells are presented in the form

$$N = Be + C\chi;\ M = Ce + Y\chi, \tag{1}$$

where *e*, χ, *N*, *M* – vectors composed of the components of membrane and bending deformations, forces and moments at an arbitrary point in the coordinate surface of a shell, which is specified by the metric within the Cartesian coordinate system, placed in terms of the shell. We consider an arbitrary contour of the shell $\omega(x, y) = 0$, lying in the plane x, y. The elements of matrices B, C, E represent integrals across the shell thickness calculated from the known function using variable parameters of elasticity:

$$\begin{gathered} \tilde{E} = \psi/(1+\psi/9K); \quad \tilde{\nu} = (1/2 - \psi/9K)/(1+\psi/9K); \\ \psi = \sigma_{VM}/\varepsilon_{VM}; \quad K = E/3(1-2\nu), \end{gathered} \tag{2}$$

where *E*, ν – the modulus of elasticity and Poisson's ratio, σ_{VM}, ε_{VM} – Mises intensity for the stresses and strains between which the functional relationship is set for the stress-strain diagram of the material of the shell.

Elastic-plastic bending of shallow shells is described by the following system of nonlinear equations [6]

$$\begin{gathered} \nabla^2\left(D_1\nabla^2 w - S_2\nabla^2\varphi\right) - \Delta_k\varphi - L(D_2, w) - \\ - L(S_1,\varphi) - L(\varphi, w) = q; \\ \nabla^2\left(S_2\nabla^2 w + H_1\nabla^2\varphi\right) + \Delta_k w + L(S_1, w) - \\ - L(H_2,\varphi) + \tfrac{1}{2}L(w,w) = 0, \end{gathered} \tag{3}$$

where

$$L(\eta,\beta) = \eta_{,11}\beta_{,22} + \eta_{,22}\beta_{,11} - 2\eta_{,12}\beta_{,12};$$

$$H_1 = b_{11}/\left(b_{11}^2 - b_{12}^2\right); \quad H_2 = \tfrac{1}{2}b_{33}; \ S_1 = c_{33}/b_{33};$$

$$S_2 = \left(b_{11}c_{12} - b_{12}c_{11}\right)/\left(2b_{33}\right);$$

$$D_1 = Y_{11} - \frac{b_{12}\left(c_{11}^2 - c_{12}^2\right) - 2c_{11}b_{11}c_{12}}{\left(b_{11}^2 - b_{12}^2\right)}; \quad D_2 = Y_{11} - Y_{12} - \frac{2c_{33}^2}{b_{33}}.$$

2 Problem-Solving Method

After linearization of the nonlinear system of equations (3) by the method of subsequent loading [8], while maintaining the same step of the loading $\widetilde{E}=\widetilde{E}_{k-1},\widetilde{\nu}=\widetilde{\nu}_{k-1}$ the system of equations (3) is transformed to the following:

$$\begin{gathered}\nabla^2\left(D_1\nabla^2\dot{w}_{(k)}-S_2\nabla^2\dot{\varphi}_{(k)}\right)-\Delta_k\dot{\varphi}_{(k)}-L\left(D_2,\dot{w}_{(k)}\right)-\\-L\left(S_1,\dot{w}_{(k)}\right)-L\left(\dot{\varphi}_{(k)},w_{(k-1)}\right)-L\left(\varphi_{(k-1)},\dot{w}_{(k)}\right)=Q;\\\nabla^2\left(S_2\nabla^2\dot{w}_{(k)}+H_1\nabla^2\dot{\varphi}_{(k)}\right)+\Delta_k\dot{w}_{(k)}+L\left(S_1,\dot{w}_{(k)}\right)+\\+L\left(H_2,\dot{\varphi}_{(k)}\right)-L\left(\dot{w}_{(k)},w_{(k-1)}\right)=0,\end{gathered}\tag{4}$$

To verify the values of stresses due to transverse strains and non-zero deformations ε_z ($\sigma_z=0$) iterative formula is used [9]:

$$\begin{gathered}\varepsilon_{z(k-1)}^{(m)}=-\frac{1-\alpha^{(m)}}{1+2\alpha^{(m)}}\left[\left(e_{1(k-1)}+e_{2(k-1)}\right)+z\left(\chi_{1(k-1)}+\chi_{2(k-1)}\right)\right];\\\alpha^{(m)}=\frac{2}{9}\frac{E_c^{(m)}}{K};\quad K=\frac{E}{3(1-2\nu)};\quad E_c^0=3G=\frac{3}{2}\frac{E}{(1+\nu)};\\C^{(m)}=1+\frac{3}{\left(1+2\alpha^{(m)}\right)^2};\quad D^{(m)}=1-\frac{3}{\left(1+2\alpha^{(m)}\right)^2};\\\varepsilon_{u(k-1)}^{(m)}=\tfrac{2}{\sqrt{3}}\sqrt{P_\varepsilon^{(m)}+2zP_{\varepsilon\chi}^{(m)}+z^2P_\chi^{(m)}};\\P_\varepsilon^{(m)}=\frac{1}{4}\left[C^{(m)}\left(e_{1(k-1)}^2+e_{2(k-1)}^2\right)-2D^{(m)}e_{1(k-1)}e_{2(k-1)}+\gamma_{(k-1)}^2\right];\\P_{\varepsilon\chi}^{(m)}=\frac{1}{4}\left[C^{(m)}\left(\chi_{1(k-1)}e_{1(k-1)}+\chi_{2(k-1)}e_{2(k-1)}\right)-\right.\\\left.-D^{(m)}\left(\chi_{21(k-1)}e_{1(k-1)}+\chi_{1(k-1)}e_{2(k-1)}\right)+2\chi_{12(k-1)}\gamma_{(k-1)}\right];\\P_\chi^{(m)}=\frac{1}{4}\left[C^{(m)}\left(\chi_{1(k-1)}^2+\chi_{2(k-1)}^2\right)+2D^{(m)}\chi_{1(k-1)}\chi_{2(k-1)}+4\chi_{12(k-1)}^2\right];\\E_c^{(m+1)}=\frac{\sigma_{VM}\left(\varepsilon_{VM}^{(m)}\right)}{\varepsilon_{VM}^{(m)}};\quad m=0,1,...M.\end{gathered}\tag{5}$$

The iterative process, described by the formula (5), is completed when the following inequality is satisfied $\left|\varepsilon_{u(k-1)}^{(m+1)}-\varepsilon_{u(k-1)}^{(m)}\right|/\varepsilon_{u(k-1)}^{(m)}<\delta$, where δ – prescribed quantity that determines the accuracy of calculations.

The system of equations (4) for each k-th loading step is the set of stationarity equations for the following variational functional

$$\begin{gathered}I_{(k)}=\int_S\left\{\frac{1}{2}D_1\left(\dot{w}_{(k),11}^2+\dot{w}_{(k),22}^2\right)+\left(D_1-D_2\right)\dot{w}_{(k),11}\dot{w}_{(k),22}+\right.\\+D_2\dot{w}_{(k),12}^2-\left(S_1+S_2\right)\left(\dot{\varphi}_{(k),22}\dot{w}_{(k),11}+\dot{\varphi}_{(k),11}\dot{w}_{(k),22}\right)-\\-S_2\left(\dot{\varphi}_{(k),11}\dot{w}_{(k),11}+\dot{\varphi}_{(k),22}\dot{w}_{(k),22}\right)+\\+2S_1\dot{\varphi}_{(k),12}\dot{w}_{(k),12}-\frac{1}{2}H_1\left(\dot{\varphi}_{(k),22}^2+\dot{\varphi}_{(k),11}^2\right)-\\-\left(H_1-H_2\right)\dot{\varphi}_{(k),11}\dot{\varphi}_{(k),22}-H_2\dot{\varphi}_{(k),12}^2-\\-\left(k_1\dot{\varphi}_{(k),22}+k_2\dot{\varphi}_{(k),11}\right)\dot{w}_{(k)}-\dot{\varphi}_{(k),11}\dot{w}_{(k),22}w_{(k-1)}-\\-\dot{\varphi}_{(k),22}\dot{w}_{(k),11}w_{(k-1)}+2\dot{\varphi}_{(k),12}\dot{w}_{(k),12}w_{(k-1)}-\\-\frac{1}{2}\dot{w}_{(k)}\dot{w}_{(k),22}\varphi_{(k-1),11}-\frac{1}{2}\dot{w}_{(k)}\dot{w}_{(k),11}\varphi_{(k-1),22}+\\\left.+\frac{1}{2}\dot{w}_{(k)}\dot{w}_{(k),12}\varphi_{(k-1),12}-Q\frac{1}{2}\dot{w}_{(k)}\right\}dS.\end{gathered}\tag{6}$$

It should be noted that under repeated use of the subsequent loading method an error caused by discarding the nonlinear terms in (4) is accumulated. Therefore, after a certain number of steps the Newton–Kantorovich method is used within the solution [10]. The initial approximation for the linearized via the Newton–Kantorovich method system of differential equations is the solution obtained using the method of subsequent loading. The accumulated experience of solving a large number of tasks on the proposed algorithm allows making a conclusion about its effectiveness, versatility and fast convergence.

After linearization of system (3) PHM we come to equations of the form

$$\begin{gathered}\nabla^2\left(D_{1n}\nabla^2W_{n+1}-S_{2n}\nabla^2\Phi_{n+1}\right)-\Delta_k\Phi_{n+1}-\\-L\left(D_{2n},W_{n+1}\right)-L\left(S_{1n},\Phi_{i+1}\right)-\\-L\left(W_n,\Phi_{n+1}\right)-L\left(\Phi_n,W_{n+1}\right)=q+L\left(W_n,\Phi_n\right);\\\nabla^2\left(S_{2n}\nabla^2W_{n+1}+H_{1n}\nabla^2\Phi_{n+1}\right)+\Delta_kW_{n+1}+\\+L\left(S_{1n},W_{n+1}\right)-L\left(H_{2n},\Phi_{n+1}\right)+\\+L\left(W_n,W_{n+1}\right)+1/2\,L\left(W_n,W_n\right)=0,\end{gathered}\tag{7}$$

where n is the iteration number.

The differential equations (7) are equivalent to the condition of stationarity of the following functional

$$\begin{gathered}K^{(n+1)}\left(w_{n+1},\varphi_{n+1}\right)=\iint_S\left\{\frac{1}{2}D_1^n\left(w_{n+1,xx}^2+w_{n+1,yy}^2\right)+\right.\\+\left(D_1^n-D_2^n\right)w_{n+1,xx}w_{n+1,yy}+D_2^nw_{n+1,xy}^2-\\-\left(S_1^n+S_2^n\right)w_{n+1,yy}\varphi_{n+1,xx}-\left(S_1^n+S_2^n\right)w_{n+1,xx}\varphi_{n+1,yy}+\\+2S_1^n\varphi_{n+1,xy}w_{n+1,xy}-S_2^n\left(w_{n+1,yy}\varphi_{n+1,yy}+w_{n+1,xx}\varphi_{n+1,xx}\right)-\\-\frac{1}{2}H_1^n\left(\varphi_{n+1,xx}^2+\varphi_{n+1,yy}^2\right)-\left(H_1^n-H_2^n\right)\varphi_{n+1,xx}\varphi_{n+1,yy}-\\-\frac{1}{2}\left(\varphi_{n+1,y}w_{n+1,y}w_{n,xx}+\varphi_{n+1,x}w_{n+1,x}w_{n,yy}-\right.\\\left.-\left(\varphi_{n+1,x}w_{n+1,y}+\varphi_{n+1,y}w_{n+1,x}\right)w_n\right)-\\-\left[\left(q-\varphi_{n,yy}w_{n,xx}-\varphi_{n,xx}w_{n,yy}\right)w_{n+1}-\right.\\\left.\left.-\left(w_{n,yy}w_{n,xx}-w_{n,xy}^2\right)\varphi_{n+1}\right]\right\}dS.\end{gathered}\tag{8}$$

Thus, the original nonlinear boundary value problem for the system (3) is reduced to solving variational problems of finding the minimum of the functionals (6) and (8). Discretization of the given functional should be carried out on the set of kinematically admissible functions satisfying the given boundary conditions. To meet the given boundary conditions the solution structure using the theory of R-functions [4] is built. The developed algorithm is realized within the POLE-RL programming system.

3 Numerical Results

This paper studied the elastic-plastic deformation of square spherical shallow shells with a radius of curvature $\overline{k}=\overline{k}_1=\overline{k}_2=18$ for the case of the simply supported boundaries conditions:

$$\Phi\big|_\Gamma=\left.\frac{\partial^2\Phi}{\partial n^2}\right|_\Gamma=0;\quad w\big|_\Gamma=\left.\frac{\partial^2w}{\partial n^2}\right|_\Gamma=0.$$

Basic physically–mechanical characteristics of the material are taken for the diagram corresponding to the linear-hardening materials, however, for different values of the yield stress. The numerical results, as a function of the central deflection load setting, are shown in Figure 1.

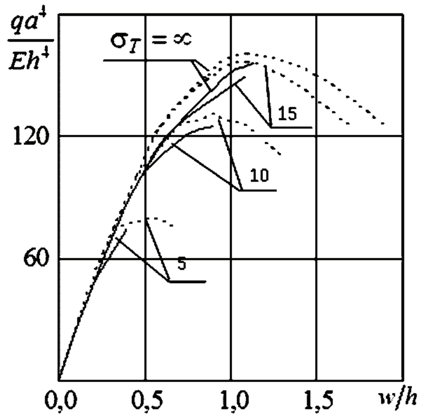

Figure 1 – Diagram "load – deflection" spherical shallow shells

From the presented data we can draw the following conclusions. If the shell material in the considered range of loading has an unlimited elasticity ($\sigma_T = \infty$), the stiffness is significantly higher than in the case of limited elasticity. With a decrease in the yield strength of the material with linear hardening, the rigidity of the shell is markedly reduced. For comparison, the dashed line in Figure 1 shows known data of work [11]. It is seen that the numerical results coincidence is quite satisfactory, which proves the above laws of the influence of plastic deformation on the bending of thin shallow shells with large displacements.

Similar conclusions follow from studies done for thin rectangular plates. The diagram "load – deflection" in this case is shown in Figure 2. Here points (x) denote data from [11].

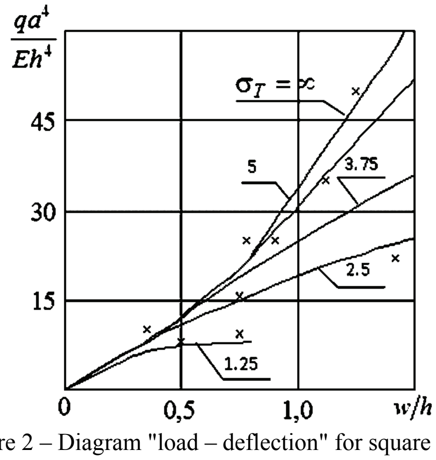

Figure 2 – Diagram "load – deflection" for square plates

In the above examples the question of the formation of plasticity zones on the surfaces of the shell and plate with proportionally increasing pressure is studied. For comparison, in Figure 3, taking into account the axial symmetry, given the plasticity zone for shells with parameters: $\overline{\sigma}_т = 15$, $\overline{E} = 0{,}01$. The plasticity zone are shown on the surface $z = \pm h/2$ when the central deflection $w = h$ and $w = 1{,}6h$. It is seen that on the surface $z = h/2$ this zone is located near the corner points of the plan, and with the deflection increase it is expanding. On the inner surface $z = -h/2$ a local zone of plasticity is forming at $w = h$ in the vicinity of the center of the shell, and as you increase the load it extends into the central part of the shell. When $w = 1{,}6h$ it covers the whole central part of this surface (Figure 3).

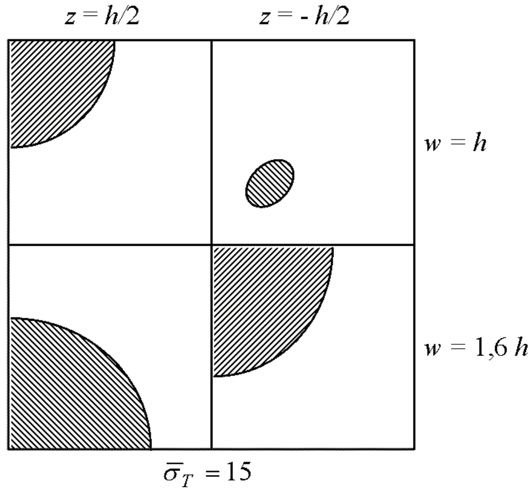

Figure 3 – Location of the plastic zones on the surface of spherical shallow shell

The formation of plastic zones in the plate is markedly different from what was observed for the shell. Calculated data of areas of plasticity in the plate of material: $\overline{\sigma}_т = 5$, $\overline{E} = 0{,}01$, on the surface $z = \pm h/2$, when the central deflection $w = h$ and $w = 1{,}6h$, is shown in Figure 4. It is seen that on the outer surface $z = h/2$ the plastic zone formed in the center of the plate and spreads with the load increase up to the corner region of the surface. However, on the inner surface these areas are significantly larger than on the outer one. Still their distribution is almost no different from that of the outer surface of the plate.

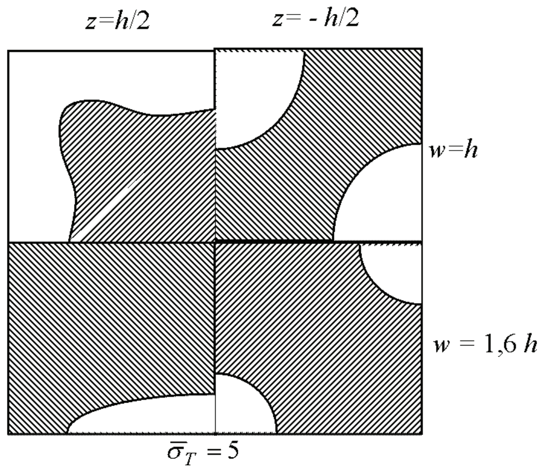

Figure 4 – The location of the plastic zones on the surface of the plate

Conclusions

The paper proposes a method to solve geometrically nonlinear elasto-plastic bending problems for thin shallow shells and plates. This method is based on the R-function

method. The solution is obtained by the Newton–Kantorovich method of subsequent loading. The developed algorithm and software are used to solve a number of test problems and to study complex-shaped shells. The paper investigates the effect of the shape of shells, boundary conditions and the load distribution on the elasto-plastic bending behavior of thin shallow shells and plates. Accumulated experience of solving a large number of tasks using the proposed algorithm allows to conclude its effectiveness, versatility and fast convergence.

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Відомості про авторів / Сведения об авторах / About the Authors

Morachkovska Irina – Candidate of Technical Sciences (Ph. D.), Docent, Associate Professor at the Department of Applied mathematics, NTU "KhPI", tel.: (050) 2983683, e-mail: i.morachkovska@gmail.com

Морачковська Ірина Олегівна –кандидат технічних наук, доцент кафедри прикладної математики, НТУ «ХПІ», tel.: (050) 2983683, e-mail: i.morachkovska@gmail.com

Морачковская Ирина Олеговна –кандидат технических наук, доцент кафедры прикладной математики, НТУ «ХПИ», tel.: (050) 2983683, e-mail: i.morachkovska@gmail.com

Timchenko Galina – Candidate of Technical Sciences (Ph. D.), Docent, Associate Professor at the Department of Applied mathematics, NTU "KhPI", e-mail: gntimchenko2000@gmail.com

Тимченко Галина Миколаївна – кандидат технічних наук, доцент кафедри прикладної математики, НТУ «ХПІ», e-mail: gntimchenko2000@gmail.com

Тимченко Галина Николаевна – кандидат технических наук, доцент кафедры прикладной математики, НТУ «ХПИ», e-mail: gntimchenko2000@gmail.com

Lyubitskaya Ekaterina – Candidate of Technical Sciences (Ph. D.), Senior Lecturer at the Department of Applied mathematics, NTU "KhPI", e-mail: lyubitska@mail.ua

Любицька Катерина Ігоревна – кандидат технічних наук, старший викладач кафедри прикладної математики, НТУ «ХПІ», e-mail: lyubitska@mail.ua

Любицкая Екатерина Игоревна – кандидат технических наук, старший преподаватель кафедры прикладной математики, НТУ «ХПИ», e-mail: lyubitska@mail.ua